Dimensional Crossover in a Spin-imbalanced Fermi Gas

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\[ \hat{H}_{1D} = \int dz \left[ \sum_{\sigma = \uparrow, \downarrow} \hat{\psi}^{\dagger}_\sigma(z) \left( \hat{H}^{sp} - \mu_\sigma \right) \hat{\psi}_\sigma(z) + g_{1D} \hat{\psi}^{\dagger}_\uparrow(z) \hat{\psi}^{\dagger}_\downarrow(z) \hat{\psi}_\downarrow(z) \hat{\psi}_\uparrow(z) \right] \]

- Large FFLO region (strong nesting)
- No long-range order
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- No long-range order
- Interactions characterized by \(1/(n a_{1D})\)
Phase diagram in 1D : Bethe Ansatz

\[ \hat{H}_{1D} = \int dz \left[ \sum_{\sigma = \uparrow, \downarrow} \hat{\psi}_{\sigma}^{\dagger}(z)(\hat{H}_{\text{sp}} - \mu_\sigma)\hat{\psi}_{\sigma}(z) + g_{1D}\hat{\psi}_{\uparrow}^{\dagger}(z)\hat{\psi}_{\downarrow}^{\dagger}(z)\hat{\psi}_{\downarrow}(z)\hat{\psi}_{\uparrow}(z) \right] \]

- Large FFLO region (strong nesting)
- No long-range order
- Interactions characterized by \(1/(na_{1D})\) → negative slope
Phase diagram in 3D: Mean field

\[
\hat{H} = \int d^3 r \left[ \sum_{\sigma = \uparrow, \downarrow} \hat{\psi}_\sigma^\dagger(\vec{r}) (\hat{H}_{\text{sp}}^\dagger - \mu_\sigma) \hat{\psi}_\sigma(\vec{r}) + g \hat{\psi}_\uparrow^\dagger(\vec{r}) \hat{\psi}_\downarrow^\dagger(\vec{r}) \hat{\psi}_\downarrow(\vec{r}) \hat{\psi}_\uparrow(\vec{r}) \right]
\]

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- Long-range order
- Positive slope
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- Small FFLO region (weak nesting)
- Long-range order
- Positive slope
- Other phases proposed: Deformed Fermi surface, Mixed phase, etc.
Why study dimensional crossover

Crossovers are interesting!

Optimal for observing FFLO (long-range order + nesting)

May give rise to new phases not present in 1D or 3D

Controllable parameters: lattice depth, densities, interaction strength
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Application: realizing the 1D model

Necessary conditions:

1. $V_0/E_R \gg 1 \implies J \to 0$ (isolated tubes)
2. Low density, and $T \ll \text{band-gap}$
   $\implies$ transverse motion frozen to the lowest energy level
Application: realizing the 1D model

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Olshanii’s mapping to an effective 1D model:

$$\frac{d_\perp \hbar \omega_\perp}{g_{1D}} = \frac{d_\perp}{2a_s} + \frac{\zeta(1/2)}{2\sqrt{2}}$$
Agrees with experiment! (near unitarity)

\[ \epsilon_B = \frac{g_1 D m}{4 \hbar^2} \]
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\[ \mu / \epsilon_B = g_1 D \frac{m}{4 \hbar^2} \]

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Assumptions:

- $J$ small $\rightarrow$ use tight-binding model
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PRL 99, 250403 (2007): $J/\varepsilon_B = 0.08$
Quasi-1D: single-band mean-field model

- 1D-like structure for all interactions for small $J$. 

![Phase diagram](image)
1D-like structure for all interactions for small $J$.
- predicts a turning point
  → not seen in experiments
  → need a more accurate model

Quasi-1D : single-band mean-field model

- Andreev bound states on each wall
- Two domain walls whose separation is large, containing two domain walls whose separation is large.
- The polarized normal (NP) phase, where the excess up spins in the ungapped, incommensurate FFLO state are not constrained.
- The gapped, commensurate FFLO state is further reduced, the 3D regime becomes sucking.
- This slice corresponds to the multicritical point where the SF to FFLO transition is continuous.
- The filled circle marks the tricritical point; near it, but not visible here is a tiny region of 'commensurate' (C) and ungapped 'incommensurate' (IC) phases.
- The SF-FFLO transition (solid line) is preempted by a first-order SF-to-NP transition.
- The large circle marks the region of FFLO where the intensity of the 2D optical lattice.
- In 1D the spectrum of BdG quasiparticles is fully gapped in the SF phase between FFLO and NP is diminished. In the limit of smaller, with the reentrance of the SF phase moving to wards 1D. Here, the FFLO phase appears and occupies a hard to detect thin shell.
- The thinness of this shell results from the small range of magnetization motion of the half of the Fermi surface to FFLO.
- The phases shown include the unpolarized SF, partially-polarized normal (N), and fully-polarized normal (NP). The FFLO phase is divided into gapped phases. The filled circle marks the tricritical point; near it, but not visible here is a tiny region of 'commensurate' (C) and ungapped 'incommensurate' (IC) phases.
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What we did

- Consider a single tube - model as a cylindrical harmonic trap
- $1D \rightarrow 3D$ crossover happens as density (or $\mu$) increases
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- Consider a single tube - model as a cylindrical harmonic trap
- 1D → 3D crossover happens as density (or $\mu$) increases
- Find mean-field phase diagram as a function of $a_s$ and $T$
- Map to an effective 1D model for $\mu < 2\hbar\omega_\perp$
  → density corrections to Olshanii’s mapping
Setting up the equations

\[ \hat{H} = \int d^3 r \left[ \sum_{\sigma = \uparrow, \downarrow} \hat{\psi}_{\sigma}^\dagger (\vec{r}) \left( \hat{H}^{\text{sp}} - \mu_{\sigma} \right) \hat{\psi}_{\sigma} (\vec{r}) + g \ \hat{\psi}_{\uparrow}^\dagger (\vec{r}) \hat{\psi}_{\downarrow}^\dagger (\vec{r}) \hat{\psi}_{\downarrow} (\vec{r}) \hat{\psi}_{\uparrow} (\vec{r}) \right] \]

where \( \hat{H}^{\text{sp}} = -\hbar^2 \nabla^2 / (2m) + (1/2)m \omega_{\perp}^2 (x^2 + y^2) \).
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Define \( \Delta(\vec{r}) = g \langle \hat{\psi}_\downarrow(\vec{r}) \hat{\psi}_\uparrow(\vec{r}) \rangle \)
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Diagonalize the BdG Hamiltonian:

\[ \hat{H}^{\text{MF}} = \sum_n \left[ (E_n-h)\hat{\gamma}_n^\dagger \hat{\gamma}_n^\uparrow + (E_n+h)\hat{\gamma}_n^\dagger \hat{\gamma}_n^\downarrow + (\varepsilon_n-E_n) \right] - \frac{1}{g} \int d^3r |\Delta(\vec{r})|^2 \]

where

\[
\begin{pmatrix}
\hat{H}^{\text{sp}} - \mu & \Delta(\vec{r}) \\
\Delta^*(\vec{r}) & \mu - \hat{H}^{\text{sp}}
\end{pmatrix}
\begin{pmatrix}
u(\vec{r}) \\ u(\vec{r})
\end{pmatrix} = E
\begin{pmatrix}
u(\vec{r}) \\ u(\vec{r})
\end{pmatrix}, \quad E_n \geq 0
\]
Regularization

Ground state energy \(( T = 0)\):

\[
\mathcal{E} = \sum_n \left[ \alpha(E_n - h) + \varepsilon_n - E_n \right] - \frac{1}{g} \int d^3 r |\Delta(\vec{r})|^2 .
\]
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\]

\(g\) is related to \(a_s\):

\[
\frac{1}{g} = \frac{m}{4\pi\hbar^2 a_s} - \int \frac{d^3k}{(2\pi)^3} \frac{m}{\hbar^2 k^2}.
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\]

For large \(n\), \(|\varepsilon_n - E_n| \ll \varepsilon_n \Rightarrow\) use perturbation theory

\[
\Rightarrow E = E_{\text{exact}} - \sum_n' \langle n|\hat{\Delta}\hat{\Delta}^\dagger|n\rangle/(2\varepsilon_n) - \frac{1}{g} \int d^3 r |\Delta(\vec{r})|^2
\]

\(\text{divergences cancel out}\)
Ansatz for $\Delta(\vec{r})$

$$\Delta(\vec{r}) = \Delta_0 \ e^{-(x^2+y^2)/\xi^2} \ e^{iqz}$$

- Variational parameters: $\Delta_0, \xi, q$

Allowed states: FF, BCS ($q = 0$), Normal ($\xi = 0$), and breached-pair ($q = 0$)
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- Variational parameters: $\Delta_0$, $\xi$, $q$
- Allowed states: FF, BCS ($q = 0$), Normal ($\Delta_0 = 0$), and breached-pair ($q = 0$)
- LO ansatz yields very similar results
Breached-pair state

What it is: a coherent mixture of Cooper pairs and unpaired fermions, which occupy different regions in momentum-space.
Breached-pair state

**What it is**: a coherent mixture of Cooper pairs and unpaired fermions, which occupy different regions in momentum-space.

**Example dispersions in 1D**:

![Graph showing dispersions in 1D](image-url)
Phase diagram at weak interactions \((a_s = -d_\perp/3)\)

1D-like structure that repeats as new channels open

\[ \epsilon_B = \frac{g_{1D}^2 m}{4 \hbar^2} \]
Change with stronger interactions ($a_s = -2d_\perp /3$)

Crossover to 3D happens at a lower density ($\mu / \hbar \omega_{\perp}$)

The SF region grows

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Change with stronger interactions \((a_s = -2d_{\perp}/3)\)

- Crossover to 3D happens at a lower density \((\mu)\)
- The SF region grows
At unitarity

- Stable BP phase emerges
- 3D-like for all densities
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- Experiment finds 1D-like behavior at low densities ($\mu \sim 1.1\hbar\omega_\perp$)

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At unitarity

- Stable BP phase emerges
- 3D-like for all densities
- Experiment finds 1D-like behavior at low densities ($\mu \sim 1.1\hbar\omega_\perp$)
- DFT produces 3D-like behavior (+BP)!
At unitarity

Does not agree with mean-field results with Olshanii’s mapping
Phase diagrams

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FF and BP dispersions

Different curves denote different transverse modes
Degenerate 2nd-order perturbation theory:

\[
\frac{d \hbar \omega \downarrow}{g_{1D}} = f\left(\frac{a_s}{d \downarrow}, \frac{\mu}{\hbar \omega \downarrow}, \frac{\Delta_0}{\hbar \omega \downarrow}, \frac{\xi}{d \downarrow}, q d \downarrow\right)
\]
Degenerate 2nd-order perturbation theory:

\[
\frac{d_\perp \hbar \omega_\perp}{g_{1D}} = f\left( \frac{a_s}{d_\perp}, \frac{\mu}{\hbar \omega_\perp}, \frac{\Delta_0}{\hbar \omega_\perp}, \frac{\xi}{d_\perp}, qd_\perp \right)
\]

Consider the limit \( \Delta_0, q \to 0, \frac{\xi}{d_\perp} \to 1 \):

\[
\frac{1}{\tilde{g}_{1D}} = \frac{1}{2\tilde{a}_s} + \frac{\zeta\left(\frac{1}{2}, 2 - \tilde{\mu}\right)}{2\sqrt{2}}
\]

\[
- \frac{\sqrt{2}}{\pi} \Theta(\tilde{\mu} - 1) \sum_{j=1}^{\infty} \frac{2^{-2j}}{\sqrt{j+1 - \tilde{\mu}}} \tan^{-1} \sqrt{\frac{\tilde{\mu} - 1}{j+1 - \tilde{\mu}}},
\]

As \( \mu \to \hbar \omega_\perp, 1/\tilde{g}_{1D} = 1/(2\tilde{a}_s) + \zeta(1/2)/(2\sqrt{2}) \) (Olshanii!)
Comparisons of effective models

Red → Mean-field with our mapping
Blue → Mean-field with Olshanii’s mapping
Green → Bethe Ansatz with our mapping
Effect of temperature

\[ \beta \hbar \omega \]