Numerical Exercises on Electromagnetism

1. Motion in the field of a random set of point charges

Consider two positive and three negative point charges of the unit magnitude, fixed at five random locations within the square $-1 < x, y < 1$.

(a) Make a density plot of the net electric potential produced by the charges in the x-y plane, with equipotential contours.

(b) Show the electric field lines on the density plot.

(c) Plot the trajectory of a test charge of magnitude +1 released from rest at the origin. Does it follow an electric field line? Briefly explain why or why not.

(d) Now assume the test charge is also experiencing a viscous drag of the form $\vec{F}_d = -C_d\vec{v}$ where C_d is the drag coefficient and $\vec{v}$ is its velocity. Repeat part (c) with $C_d = 0.1$, 1, 10, and 100! How does the trajectory change with larger drag? (Superimpose each trajectory on the electric field lines to see the difference.) Do you understand why?

(e) Plot the total energy of the test charge as a function of time for $C_d = 0$, 0.1, 1, and 10. Label your axes, and plot the four cases with different colors, e.g., black, blue, red, and green. Finally, show them together, adjusting the plot range for comparison.

(f) Explain your findings - is the total energy conserved? Why or why not? What is the effect of increasing drag? Does it make sense?

2. Simulating a conductor

In section we saw that a bunch of identical point charges confined within a circle does *not* end up on its periphery under mutual repulsion. Let's find out what happens in 3D.

Say we have 30 point charges, all having unit charge and unit mass, confined within a sphere of radius $R = 3$. The charges are free to move within the sphere, and bounce elastically off the walls. You can model the wall by a steep potential, e.g. by $V(r) = e^{100*(r-3)}$. Suppose the charges are released at rest from random locations inside the cube $-1 < x, y, z < 1$.

(a) Numerically solve their equations of motion for $0 < t < 50$. Make a movie of how they move inside the sphere! *Hint:* use "Graphics3D" and "Manipulate" in Mathematica. You can control the opacity of the sphere by using the option "Opacity[p]," where $0 < p < 1$.

(b) To quantify the spread of the charges, plot their root-mean-squared distance vs time, $\sqrt{< r^2(t) >}$, where the averaging is over all particles. Do they all end up on the surface? What value does $\sqrt{< r^2(t) >}$ approach at long times? How does it compare with the case where the charges are uniformly distributed inside the sphere?

(c) If they don't end up on the surface, does that contradict what we know about conductors? What are the conserved quantities here - energy? momentum? angular momentum?

(d) Now assume the charges also experience a drag force $\vec{F}_d = -0.2\,\vec{v}$. Repeat (a) and (b). (*Think:* is there an equivalent of a drag force in real conductors?)

(e) Do you get different results from what we saw in 2D? If so, why is 2D and 3D so different? *Interesting reading:* (i) Fun with electrostatics, (ii) The charged conducting disk, (iii) http://www.physics.umd.edu/grt/taj/411b/AJP000155.pdf.

3. Motion in a Penning trap

Many labs store charges particles in a Penning trap (e.g., CERN store antimatter). Such a trap makes use of a quadruple electric field described by the potential $\Phi(\vec{r}) = \frac{V_0}{2}\frac{z^2-(x^2+y^2)/2}{d^2}$, and an axial homogenous magnetic field $\vec{B}(\vec{r}) = B_0\hat{z}$. Here d is the spatial extent of the trap, and V_0, B_0 are field strengths.

(a) Write the equations of motion of a particle with charge q and mass m in such a trap in Cartesian coordinates, in terms of the frequencies $\omega_c \equiv \frac{qB_0}{mc}$ and $\omega_z \equiv \sqrt{\frac{qV_0}{md^2}}$. Non-dimensionalize your equations by rescaling time by ω_z (i.e., define $\tau \equiv \omega_z t$, and write the equations in terms of $\frac{d}{d\tau}$ instead of $\frac{d}{dt}$).

(b) Analytically solve the motion along $\hat{z}$.

(c) Show that $E = \dot{x}^2 + \dot{y}^2 - \frac{1}{2}(x^2 + y^2)$ does not change with time, where $\dot{x} \equiv \frac{dx}{d\tau}$.

(d) For different choices of the ratio ω_c/ω_z (say, 1,2,4), numerically solve the x-y motion, after the particle is released at rest from $x = 1$, $y = 0$. Plot its trajectories. Can you find the condition on ω_c/ω_z for which the particle does not leave the trap?

(e) Plot E (as defined above) along one of the trajectories to verify that it is indeed conserved.

(f) Plot the trajectory for $\omega_c/\omega_z = (n+1)/\sqrt{2n}$, with $n = 2,3,4,5$, etc. for the same initial conditions. What special feature do you see for these values?
(*Think:* Why?? What happens for $n = 1$?)

Further info about the principle and applications of Penning traps:
http://journals.aps.org/rmp/abstract/10.1103/RevModPhys.58.233
http://www.tandfonline.com/doi/abs/10.1080/00107510903387652
http://gabrielse.physics.harvard.edu/gabrielse/papers/1990/1990_tjoelker/chapter_2.pdf

4. Field of a realistic parallel-plate capacitor

Our goal is to find out how the finite size of the parallel plates distort the electric fields. For simplicity, we'll take the plates to be infinitely long in one dimension (say, z), so that we can take a cross-section in the x-y plane, and solve for the fields in that plane. The problem is effectively two-dimensional.

Thus, consider two infinitely long parallel conducting plates, each 2 units thick and 50 units wide. They are facing each other at a separation of 10 units. One plate is held at a potential of +100, and the other at −100.

(a) Numerically solve Laplace's equation to find the electric potential in a large box in the x-y plane surrounding the plates. Take the potential on surface of the large box to be zero.

The large box acts as a ground far away from the plates. It should be large enough not to affect the charges on the plates appreciably. A good estimate is twice the width of plates, i.e., a 100 unit × 100 unit box should be large enough.

(b) Make a density plot of the potential in the x-y plane, and show the electric field lines. Do they make sense? e.g., how does the electric field between the two plates compare with the field outside?

5. Solving Laplace's equation on a grid by relaxation method

Consider two infinitely long parallel conducting plates, each 2 units thick and 50 units wide. They are facing each other at a separation of 10 units. One plate is held at a potential

$V_1 = +100$, and the other at $V_2 = -25$. The program is to solve Laplace's equation on a grid in the x-y plane inside a large 100 unit × 100 unit box held at zero potential.

(a) What equations do the potentials at the grid points, Φ_{ij}, satisfy? Briefly explain why.

(b) Solve the set of equations exactly with proper boundary conditions. Make a density plot of the potential inside the large box, with 50 equipotential contours.

(c) We'll now solve the equations using the relaxation method, i.e., repeatedly updating Φ_{ij} from its neighbors after starting with an initial guess. To get a measure of the error in the potential profile at any stage, we can evaluate the quantity

$$\text{percentage error} = 100 \times \frac{\sum_{i,j}(\Phi_{ij}^{\text{relax}} - \Phi_{ij}^{\text{exact}})^2}{V_1^2 L_x L_y} \,,$$

where L_x and L_y are the dimensions of the box (both 100 in our case). You can define your own error function if you're not happy with the above!

(i) Make a density plot of the potential after 50 iterations (with 50 equipotentials). Compute the percentage error. Name these results so that you can access them later.

(ii) Repeat part (i) for 9 more rounds! Briefly explain what you observe.

(iii) Show how the error decreases with the number of iterations by a making a plot. (To plot discrete data points in Mathematica, use the function "ListPlot.")

6. Dispersion: spreading of wave-packets

Consider a wave-packet initially localized near $x = 0$, described by the waveform

$$f(x, t = 0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{ikx} e^{-k^2}.$$

As time passes, the Fourier component with wave-vector k travels at speed $v_k = \omega_k / k$, which gives rise to a phase factor $e^{-i\omega_k t}$ in the integral. Hence at finite time t, the waveform is described by

$$f(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{i(kx - \omega_k t)} e^{-k^2}.$$

Notice that this is just the Fourier transform of the function $F(k) = e^{-k^2 - i\omega_k t}$. For the following questions, calculate $f(x, t)$ by taking discrete samples of $F(k)$ from $k = -20$ to $k = +20$ in steps of 0.02.

(a) *Non-linearity:* Consider the dispersion relation $\omega_k = k + \varepsilon k^2$, where ε controls the non-linearity. Make animations showing the evolution of $|f(x,t)|$ over time from $t = 0$ to $t = 20$, for $\varepsilon = 0, 0.1, 0.2, 0.3, 0.4$.

You'll find that the peak of each waveform travels at unit speed, but the waveforms spread out at different rates. Calculate the width of each waveform about its center at $t = 20$, and plot the widths vs the parameter ε. Briefly explain what you find.

Hint: You can calculate the width by finding a location x^* where $|f(x^*)|$ is half the peak value. The function "FindRoot" can be handy in Mathematica.

(b) *Power-law:* Consider the dispersion relation $\omega_k = |k|^\alpha$. Make animations showing the evolution of $|f(x,t)|$ from $t = 0$ to $t = 20$, for $\alpha = 0.5, 1, 2, 3$. Briefly explain the kind of changes you observe with the exponent α. Do they make sense?

(c) *Uncertainty relation:* Consider the free-particle dispersion in quantum mechanics, $\omega_k = k^2$. Consider wave-packets with different widths (Δ) in k-space:

$$f(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, e^{i(kx - k^2 t)} e^{-(k/\Delta)^2}.$$

Make animations showing the evolution of $|f(x,t)|$ from $t = 0$ to $t = 4$, for $\Delta = 1, 2, 3, 4$. Plot the widths of the waveforms vs Δ at $t = 0$ and $t = 4$. Can you connect what you find to Heisenberg's uncertainty relation?

7. Random walk of solar photons

The transport of electromagnetic energy from the center to the surface of the sun can be described by a random walk model of photons: in each time-step, a photon travels a fixed distance l at speed c before being scattered in a random direction by an electron. After sufficiently many time-steps, the photon reaches the surface and leaves the sun. We'll use numerics to characterize some properties of such a random walk.

(a) Taking $l = 1$ unit, plot a random-walk trajectory in 3D for 1000 time-steps.

Next we'll find out how many steps (N) a random walker takes before reaching a distance $R = 100l$ from the origin. We'll also count how many times (n) it crosses back inside a radius $r = R/10$ during the process.

(b) Numerically find N and n for 1000 different random walks in 3D. Plot histograms showing their distributions. Use 100 bins for each histogram.

Hint: Define a function which calculates N and n for a given random walk. To calculate n, you can have a counter which is initially set to 0 and increases by 1 every time the random walker crosses inside a radius r.

(c) Calculate the mean ($\langle \ldots \rangle$) and standard deviation ($\sigma_{\ldots}$) of both N and n from your data in (b).

It is known from past studies that for large R/l, $\langle N \rangle$ and σ_N grow as $(R/l)^2$, whereas $\langle n \rangle$ and σ_n grow as R/l. One can also estimate l and R for solar photons (see, e.g., this article) as $l = 0.5$ mm and $R = 7 \times 10^5$ km.

(d) Using these info and your results in part (c), calculate $\langle N \rangle$, σ_N, $\langle n \rangle$, and σ_n for solar photons.

(e) From $\langle N \rangle$ and σ_N, estimate the time taken for a photon to reach the solar surface, assuming it travels at speed c during each step. (There are $\pi \times 10^7$ seconds in a year!)

What was happening on Earth when the sunlight that reached your eye today left the center of the sun? You may find this a good reference: spread of modern humans!

(f) Repeat (b) and (c) in 2D and 1D. What change(s) do you notice in lower dimensions? Do they make sense?

8. Detecting a magnetic monopole

One way to detect a magnetic monopole is to look for currents induced in a coil of wire by the monopole. Let's see how the scheme works.

The magnetic field produced by a monopole of magnetic charge q_m is radially outward, similar to the electric field produced by an electric charge:

$$\vec{B}(\vec{r}) = \frac{q_m}{r^2}\hat{r}.$$

Suppose we have a electric point charge q_e confined to a ring in the x-y plane. The ring itself is non-conducting. The point charge is initially stationary (i.e., at $t \to -\infty$). A magnetic monopole of charge q_m travels along the z axis at a uniform velocity $\vec{v} = v\hat{z}$ ($v > 0$), crossing the center of the ring at $t = 0$.

(a) Argue that the passage of the monopole will set the point charge in motion along the ring. Does it rotate counterclockwise or clockwise, as seen from the positive z axis (assuming $q_e q_m > 0$)?

(b) Find out the final angular momentum ($\vec{L}_{\text{final}}$) of the point charge about the axis of the ring (i.e., at $t \to \infty$). Is this a violation of the conservation of angular momentum?

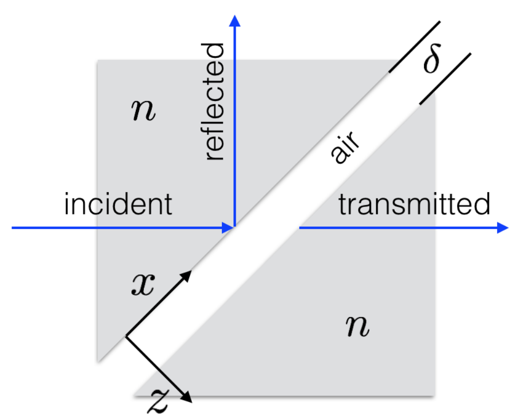
Hint: You may find it useful to work from the monopole frame.

(c) You should find that $\vec{L}_{\text{final}}$ only depends on the product $q_e q_m$. In quantum mechanics, angular momentum is quantized, i.e., it can only take equally spaced discrete values. Explain how/whether this relates to the quantization of charge.

(d) Make a movie showing the motion of the monopole and the point charge. For the animation you can set $v = 0.5$, and all other parameters (including c) to 1 (ignore relativistic corrections).

Fun fact: Such an experiment was indeed performed in the 1980s, and ended in a rather intriguing result. See this blog or this article if you want to know more!

9. Energy flow in frustrated total internal reflection



Consider the problem in the last homework on total internal reflection, as sketched in the figure. In section we derived the electromagnetic fields in the various regions from Maxwell's equations:

$$\vec{E}_i(\vec{r},t) = \hat{y}\,E_{iy}(\vec{r},t) \qquad \vec{B}_i(\vec{r},t) = \frac{\hat{z}-\hat{x}}{\sqrt{2}}\,n\,E_{iy}(\vec{r},t)$$

$$\vec{E}_r(\vec{r},t) = \hat{y}\,E_{ry}(\vec{r},t) \qquad \vec{B}_r(\vec{r},t) = \frac{\hat{z}+\hat{x}}{\sqrt{2}}\,n\,E_{ry}(\vec{r},t)$$

$$\vec{E}_t(\vec{r},t) = \hat{y}\,E_{ty}(\vec{r},t) \qquad \vec{B}_t(\vec{r},t) = \frac{\hat{z}-\hat{x}}{\sqrt{2}}\,n\,E_{ty}(\vec{r},t)$$

$$\vec{E}_a(\vec{r},t) = \hat{y}\,E_{a+}(\vec{r},t) \qquad \vec{B}_a(\vec{r},t) = \hat{z}\,\frac{n}{\sqrt{2}}\,E_{a+}(\vec{r},t) + \hat{x}\sqrt{\frac{n^2}{2}-1}\,E_{a-}(\vec{r},t)$$

where

$$E_{iy}(\vec{r},t) = \mathrm{Re}\left\{\mathcal{E}_i \exp\left[i\left(2\pi n\frac{\tilde{x}+\tilde{z}}{\sqrt{2}} - \tilde{t}\right)\right]\right\}$$

$$E_{ry}(\vec{r},t) = \mathrm{Re}\left\{\mathcal{E}_r \exp\left[i\left(2\pi n\frac{\tilde{x}-\tilde{z}}{\sqrt{2}} - \tilde{t}\right)\right]\right\}$$

$$E_{ty}(\vec{r},t) = \mathrm{Re}\left\{\mathcal{E}_t \exp\left[i\left(2\pi n\frac{\tilde{x}+\tilde{z}-\tilde{\delta}}{\sqrt{2}} - \tilde{t}\right)\right]\right\}$$

$$E_{a\pm}(\vec{r},t) = \mathrm{Re}\left\{\left(\mathcal{E}_{a2}\,e^{2\pi\sqrt{n^2/2-1}\,\tilde{z}} \pm \mathcal{E}_{a1}\,e^{-2\pi\sqrt{n^2/2-1}\,\tilde{z}}\right)\exp\left[i\left(2\pi n\frac{\tilde{x}}{\sqrt{2}} - \tilde{t}\right)\right]\right\}$$

and

$$\mathcal{E}_i = \left[\cosh\left(2\pi\sqrt{n^2/2-1}\,\tilde{\delta}\right) - \frac{i\sinh\left(2\pi\sqrt{n^2/2-1}\,\tilde{\delta}\right)}{n\sqrt{n^2-2}}\right]\mathcal{E}_t$$

$$\mathcal{E}_r = -i\frac{n^2-1}{n\sqrt{n^2-2}}\sinh\left(2\pi\sqrt{n^2/2-1}\,\tilde{\delta}\right)\mathcal{E}_t$$

$$\mathcal{E}_{a1} = \frac{1}{2}\left[1 - i\frac{n}{\sqrt{n^2-2}}\right]e^{2\pi\sqrt{n^2/2-1}\,\tilde{\delta}}\,\mathcal{E}_t$$

$$\mathcal{E}_{a2} = \frac{1}{2}\left[1 + i\frac{n}{\sqrt{n^2-2}}\right]e^{-2\pi\sqrt{n^2/2-1}\,\tilde{\delta}}\,\mathcal{E}_t$$

In these equations, $\tilde{x} \equiv x/\lambda$, $\tilde{z} \equiv z/\lambda$, $\tilde{\delta} \equiv \delta/\lambda$, and $\tilde{t} \equiv \omega t$, where λ is the wavelength in vacuum. The subscripts i, r, t, and a stand for "incident," "reflected," "transmitted," and "air" respectively.

(a) Show that the net electromagnetic energy density in the different regions is given by

$$U(\vec{r},t) = \begin{cases} \frac{n^2}{4\pi}\left(E_{iy}^2 + E_{ry}^2 + E_{iy}E_{ry}\right) & z<0 \\ \frac{1}{8\pi}\left[\left(\frac{n^2}{2}+1\right)E_{a+}^2 + \left(\frac{n^2}{2}-1\right)E_{a-}^2\right] & 0<z<\delta \\ \frac{n^2}{4\pi}E_{ty}^2 & z>\delta \end{cases}$$

For the following questions, the energy distribution refers to a density plot of $U(\vec{r}, t)$ in the x-z plane for $-1.5 < z/\lambda < 2$ and $0 < x/\lambda < 3.5$, with $\mathcal{E}_i = 1$.

(b) For $n = 1.5$ and $\delta/\lambda = 0.05$, show how the energy distribution evolves with time as ωt goes from 0 to 2π in steps of 0.1π.

Hint: First you need to define a piecewise function which computes U for given values of n, δ/λ, x/λ, z/λ, and ωt. Then you can make a table of plots for different values of ωt, and export them as a ".mov" video file. To match the orientation of the system, you can rotate each plot by -45° (in Mathematica).

(c) Repeat part (b) for $n = 1.5$ and $\delta/\lambda = 0.6$. What difference(s) do you notice?

(d) For $n = 1.5$ and $t = 0$, show how the energy distribution changes with δ/λ as it is varied from 0 to 1 in steps of 0.05. Briefly explain what you find.

(e) For $\delta/\lambda = 0.4$ and $t = 0$, show how the energy distribution changes with n as it is varied from 1.42 to 1.8 in steps of 0.02. Briefly explain what you find.