Supplement for
"Out-of-equilibrium steady states of a locally driven lossy qubit array"

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CONTENTS

As described in the main text, we consider hard-core bosons on a 1D lattice modeled by the Hamiltonian

$$\hat{H} = -\hbar J \sum_{i=1}^{L-1} \hat{b}_i^\dagger \hat{b}_{i+1} + \hat{b}_{i+1}^\dagger \hat{b}_i \;, \tag{S1}$$

where the boson operators satisfy $[\hat{b}_i, \hat{b}_j] = 0$ and $[\hat{b}_i, \hat{b}_j^\dagger] = (-1)^{\hat{n}_i} \delta_{ij}$, for site occupation $n_i \in \{0, 1\}$. The bosons are coupled to Markovian reservoirs that inject particles at site p with rate γ_+ and removes particles from site q with rate γ_-. The resulting dynamics are modeled by a master equation for the density matrix $\hat{\rho}$

$$\frac{d\hat{\rho}}{dt} = \mathcal{L}\hat{\rho} := -\frac{\mathrm{i}}{\hbar}[\hat{H}, \hat{\rho}] + \sum_{\alpha=\pm} \hat{L}_\alpha \hat{\rho} \hat{L}_\alpha^\dagger - \frac{1}{2}\{\hat{L}_\alpha^\dagger \hat{L}_\alpha, \hat{\rho}\} \;, \tag{S2}$$

where $\hat{L}_+ := \sqrt{\gamma_+}\, \hat{b}_p^\dagger$ and $\hat{L}_- := \sqrt{\gamma_-}\, \hat{b}_q$ are Lindblad operators describing the incoherent pump and loss, respectively. The system is equivalent to a spin-1/2 XX chain with local spin flips if one identifies $\hat{b}_i$ with the spin lowering operator. It can also be mapped onto fermions by a Jordan-Wigner transformation,

$$\hat{f}_j = (-1)^{\sum_{i<j} \hat{n}_i} \hat{b}_j \;, \tag{S3}$$

where $\{\hat{f}_i, \hat{f}_j\} = 0$ and $\{\hat{f}_i, \hat{f}_j^\dagger\} = \delta_{ij}$. The transformed Hamiltonian describes free fermions,

$$\hat{H} = -\hbar J \sum_{i=1}^{L-1} \hat{f}_i^\dagger \hat{f}_{i+1} + \hat{f}_{i+1}^\dagger \hat{f}_i \;. \tag{S4}$$

However, the Lindblad operators $\hat{L}_\pm$ are nonlocal in the fermions, mediating interactions.

A. CLOSED-FORM SOLUTION FOR END DRIVES

As shown in Ref. [S1] and discussed in the main text, for pump and loss at opposite ends ($p = 1$, $q = L$), the dynamics are identical to those of free fermions, i.e., with $\hat{L}_+ := \sqrt{\gamma_+}\, \hat{f}_1^\dagger$ and $\hat{L}_- := \sqrt{\gamma_-}\, \hat{f}_L$ in Eq. (S2). The steady

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state is unique [S2] and has been solved exactly in terms of a matrix product ansatz [S3]. Here we present a closed-form solution by a more direct approach.

We adopt a thermofield representation [S4, S5] where one defines a new set of operators $\tilde{f}_i$ and $\tilde{f}_i^\dagger$ that act on the density matrix by right multiplication, i.e., $\tilde{f}_i\rho := \rho f_i$ and $\tilde{f}_i^\dagger\rho := \rho f_i^\dagger$. Note we have omitted the hat for operators to reduce clutter. It is straightforward to verify the following relations for any two operators a and b:

$$[a,\tilde{b}] = 0\,, \quad (ab)^{\sim} = \tilde{b}\tilde{a}\,, \quad \{\tilde{a},\tilde{b}\} = \{a,b\}^{\sim}\,, \text{ and } [\tilde{a},\tilde{b}] = [b,a]^{\sim}. \tag{S5}$$

The Liouvillian $\mathcal{L}$ in Eq. (S2) can be expressed in this notation as

$$\mathcal{L} = \mathrm{i}J\mathcal{T} + \gamma_+\mathcal{D}_1^+ + \gamma_-\mathcal{D}_L^- \,, \tag{S6}$$

where

$$\mathcal{T} := \sum_{i=1}^{L-1} f_{i+1}^\dagger f_i + \tilde{f}_{i+1}^\dagger\tilde{f}_i + f_i^\dagger f_{i+1} + \tilde{f}_i^\dagger\tilde{f}_{i+1} \,, \tag{S7a}$$

$$\mathcal{D}_i^+ := f_i^\dagger\tilde{f}_i - \big(f_i f_i^\dagger + \tilde{f}_i^\dagger\tilde{f}_i\big)/2 \,, \tag{S7b}$$

$$\text{and} \quad \mathcal{D}_i^- := f_i\tilde{f}_i^\dagger - \big(f_i^\dagger f_i + \tilde{f}_i\tilde{f}_i^\dagger\big)/2 \,. \tag{S7c}$$

To write down the steady state, we define the generators

$$\mathcal{W}_{i,j} := [f_i^\dagger, \{f_j, \boldsymbol{\cdot}\}] = \big(f_i^\dagger - \tilde{f}_i^\dagger\big)\big(f_j + \tilde{f}_j\big) \,, \tag{S8a}$$

$$\text{and} \quad \mathcal{A} := \sum_{i=1}^{L-1} \mathcal{W}_{i+1,i} - \mathcal{W}_{i,i+1} \,. \tag{S8b}$$

We will show the steady state is given by $\rho = e^{\mathcal{G}\tau}\rho_0$, where

$$\mathcal{G} = \gamma_-\mathcal{W}_{1,1} - \gamma_+\mathcal{W}_{L,L} + 2\,\mathrm{i}J\mathcal{A} + (\gamma_- - \gamma_+)\textstyle\sum_{j=2}^{L-1}\mathcal{W}_{j,j} \,, \tag{S9}$$

$$\tau = (\gamma_+ + \gamma_-)^{-1}\big[1 + 4J^2/(\gamma_+\gamma_-)\big]^{-1} , \tag{S10}$$

and ρ_0 is a uniform product state with occupation $n_0 = \gamma_+/(\gamma_+ + \gamma_-)$,

$$\rho_0 = \prod_{i=1}^{L} n_0 f_i^\dagger f_i + (1-n_0) f_i f_i^\dagger \,. \tag{S11}$$

To prove that $\mathcal{L}\rho = 0$, we first use the identities in Eq. (S5) to find the commutators

$$[\mathcal{T},\mathcal{W}_{j,j}] = (1-\delta_{j,1})(\mathcal{W}_{j-1,j} - \mathcal{W}_{j,j-1}) + (1-\delta_{j,L})(\mathcal{W}_{j+1,j} - \mathcal{W}_{j,j+1}) \,, \tag{S12a}$$

$$[\mathcal{T},\mathcal{A}] = 2\,(\mathcal{W}_{1,1} - \mathcal{W}_{L,L}) \,, \tag{S12b}$$

$$[\mathcal{D}_i^\pm,\mathcal{W}_{j,j}] = -\delta_{i,j}\mathcal{W}_{i,i} \,, \tag{S12c}$$

$$[\mathcal{D}_i^\pm,\mathcal{A}] = [(1-\delta_{i,1})(\mathcal{W}_{i-1,i} - \mathcal{W}_{i,i-1}) - (1-\delta_{i,L})(\mathcal{W}_{i+1,i} - \mathcal{W}_{i,i+1})]/2 \,. \tag{S12d}$$

Using these results in Eqs. (S6) and (S9) yields

$$[\mathcal{L},\mathcal{G}\tau] = \frac{\gamma_+\gamma_-}{\gamma_+ + \gamma_-}(\mathcal{W}_{L,L} - \mathcal{W}_{1,1}) \,. \tag{S13}$$

Further, one can show $[\mathcal{W}_{i,i},\mathcal{G}] = 0 \;\forall i$. Thus, $[[\mathcal{L},\mathcal{G}\tau],\mathcal{G}\tau] = 0$ and $[\mathcal{L}, e^{\mathcal{G}\tau}] = e^{\mathcal{G}\tau}[\mathcal{L},\mathcal{G}\tau]$. It thereby follows that

$$\mathcal{L}\rho = e^{\mathcal{G}\tau}(\mathcal{L}\rho_0 + [\mathcal{L},\mathcal{G}\tau]\rho_0) \,. \tag{S14}$$

The expression within parentheses can be evaluated by noting that $\rho_0 = (\gamma_+/\gamma_-)^N$ up to normalization, where N is the total number operator. Hence, $[H,\rho_0] = 0$, or $\mathcal{T}\rho_0 = 0$. One also finds, using Eqs. (S11) and (S13),

$$\gamma_+\mathcal{D}_1^+\rho_0 = \big[(\gamma_+ + \gamma_-)f_1^\dagger f_1 - \gamma_+\big]\rho_0 \,, \tag{S15a}$$

$$\gamma_-\mathcal{D}_L^-\rho_0 = \big[\gamma_+ - (\gamma_+ + \gamma_-)f_L^\dagger f_L\big]\rho_0 \,, \tag{S15b}$$

$$[\mathcal{L},\mathcal{G}\tau]\rho_0 = (\gamma_+ + \gamma_-)\big(f_L^\dagger f_L - f_1^\dagger f_1\big)\rho_0 \,. \tag{S15c}$$

Substituting the above results in Eq. (S14) gives $\mathcal{L}\rho = 0$, thus showing ρ is indeed the steady state.

The computation of ρ can be simplified by noting the generators in Eq. (S9) commute with one another, $[\mathcal{W}_{i,i}, \mathcal{A}] = 0$ and $[\mathcal{W}_{i,i}, \mathcal{W}_{j,j}] = 0\ \forall\, i, j$. Thus, $e^{\mathcal{G}\tau}$ factorizes into a product of exponentials. Further, $\mathcal{W}_{i,i}$ acts locally and one can show $\mathcal{W}_{i,i}^2 = 0\ \forall i$, which means the action of these local generators on ρ_0 can be written explicitly, yielding

$$\rho = e^{2\,\mathrm{i} J\tau\mathcal{A}} \prod_{i=1}^{L} (n_0 + \Delta_i) f_i^\dagger f_i + (1 - n_0 - \Delta_i) f_i f_i^\dagger \,, \tag{S16}$$

where $\Delta_i := [(1-\delta_{i,L})\gamma_- - (1-\delta_{i,1})\gamma_+]\,\tau$. Furthermore, $\mathcal{A}^{L+1} = 0$, so the above exponential reduces to a sum of the first $L+1$ terms in its power series, which can be computed iteratively using $\mathcal{A}\,\sigma = \sum_{i=1}^{L-1} [f_{i+1}^\dagger, \{f_i, \sigma\}] - \mathrm{H.c.}$. Such a simple algebraic structure of the steady state is related to the q-deformed SU(2) symmetry of the XXZ chain [S6].

For weak dissipation ($\gamma_\pm \ll J$), $J\tau \approx \gamma_+\gamma_-/[4J(\gamma_+ + \gamma_-)] + O((\gamma_\pm/J)^3)$ and $\Delta_i \sim O((\gamma_\pm/J)^2)$. Thus, to linear order in $\gamma_\pm/J$, Eq. (S16) gives the steady state

$$\rho \approx \rho_0 + \mathrm{i}\,\frac{\gamma_+\gamma_-}{2J(\gamma_+ + \gamma_-)}\,\mathcal{A}\rho_0 = \rho_0 + \left(\mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J} \sum_{j=1}^{L-1} f_{j+1}^\dagger f_j\, \rho_0 + \mathrm{H.c.}\right), \tag{S17}$$

where we have used Eq. (S11) to simplify $\mathcal{A}\rho_0$. The perturbation to ρ_0 induces nearest-neighbor correlations,

$$\langle b_j^\dagger b_{j+1}\rangle = \langle f_j^\dagger f_{j+1}\rangle \approx \mathrm{i}\,\frac{\gamma_+\gamma_-}{2J(\gamma_+ + \gamma_-)} \,. \tag{S18}$$

For strong dissipation ($\gamma_\pm \gg J$), $J\tau \approx J/(\gamma_+ + \gamma_-) + O((J/\gamma_\pm)^3)$, and the product state in Eq. (S16) approaches ρ_Z where the first site is filled, the last site is empty, and the other sites are in a product state with occupation $1 - n_0$,

$$\rho_Z := f_1^\dagger f_1 f_L f_L^\dagger \prod_{i=2}^{L-1} (1 - n_0) f_i^\dagger f_i + n_0 f_i f_i^\dagger \,. \tag{S19}$$

To first order in $J/\gamma_\pm$, the steady state is given by

$$\rho \approx \rho_Z + \left(\mathrm{i}\,\frac{2J}{\gamma_+ + \gamma_-} \sum_{j=1}^{L-1} [f_{j+1}^\dagger, \{f_j, \rho_Z\}] + \mathrm{H.c.}\right), \tag{S20}$$

which again yields nearest-neighbor correlations,

$$\langle b_j^\dagger b_{j+1}\rangle = \langle f_j^\dagger f_{j+1}\rangle \approx \mathrm{i}\,\frac{2J}{\gamma_+ + \gamma_-} \,. \tag{S21}$$

Hence, the correlations are limited to nearest neighbors in both limits, and are purely imaginary, which corresponds to a probability current from the source at the first site to the sink at the last site.

B. PERTURBATIVE SOLUTIONS FOR A DIPOLE DRIVE

In Sec. III of the main text, we described a "dipole" arrangement of the pump and loss, where increasing dissipation establishes long-range coherence, in sharp contrast to the end-driven case studied above. In this "dipole" setup, the pump and loss occur at neighboring sites in the middle, $p = L/2$ and $q = L/2 + 1$ for even L. Here we derive the perturbation results for weak and strong dissipation quoted in the main text.

The Liouvillian $\mathcal{L}$ in Eq. (S2) can be restated as

$$\mathcal{L} = -(\mathrm{i}/\hbar)[\hat{H}, \cdot\,] + \gamma_+ \mathcal{D}[\hat{b}_{L/2}^\dagger] + \gamma_- \mathcal{D}[\hat{b}_{L/2+1}] \,, \tag{S22}$$

where $\mathcal{D}[\hat{x}]\hat{\rho} := \hat{x}\hat{\rho}\hat{x}^\dagger - \{\hat{x}^\dagger\hat{x}, \hat{\rho}\}/2$. The steady state at weak dissipation approaches the product state $\hat{\rho}_0 \propto (\gamma_+/\gamma_-)^{\hat{N}}$, as in the end-driven geometry [see Eq. (S11)]. This is the zeroth order solution, which commutes with the Hamiltonian. The solution to first order in $\gamma_\pm/J$ is given by $\hat{\rho}_\mathrm{w} \approx \hat{\rho}_0 + \hat{\rho}_1$, such that

$$-(\mathrm{i}/\hbar)[\hat{H}, \hat{\rho}_1] + \gamma_+ \mathcal{D}[\hat{b}_{L/2}^\dagger]\hat{\rho}_0 + \gamma_- \mathcal{D}[\hat{b}_{L/2+1}]\hat{\rho}_0 = 0 \,. \tag{S23}$$

We will show this is satisfied by

$$\hat{\rho}_1 = \mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J}\,(\hat{Q} - \hat{Q}^\dagger)\hat{\rho}_0 \,, \tag{S24}$$

$$\text{where}\quad \hat{Q} := \sum_{k=1}^{L/2} \hat{f}_{L+1-k}^\dagger \hat{f}_k \,. \tag{S25}$$

First, we calculate the action of the dissipators on the unperturbed solution. As in Eqs. (S15a) and (S15b), we find

$$\gamma_+\mathcal{D}[\hat{b}^\dagger_{L/2}]\hat{\rho}_0 + \gamma_-\mathcal{D}[\hat{b}_{L/2+1}]\hat{\rho}_0 = (\gamma_+ + \gamma_-)(\hat{n}_{L/2} - \hat{n}_{L/2+1})\hat{\rho}_0 \ . \tag{S26}$$

Next, we find the commutator $[\hat{H}, \hat{\rho}_1]$ using Eq. (S4) and the identity $[ab, cd] = a\{b, c\}d + ca\{b, d\} - \{a, c\}bd - c\{a, d\}b$,

$$-(\mathrm{i}/\hbar)[\hat{H}, \hat{\rho}_1] = -\frac{\gamma_+ + \gamma_-}{2}\Big(\sum_{i=1}^{L-1}[\hat{f}^\dagger_i\hat{f}_{i+1} + \hat{f}^\dagger_{i+1}\hat{f}_i, \hat{Q}] + \mathrm{H.c.}\Big)\hat{\rho}_0 = (\gamma_+ + \gamma_-)(\hat{n}_{L/2+1} - \hat{n}_{L/2})\hat{\rho}_0 \ . \tag{S27}$$

Combining Eqs. (S26) and (S27) readily gives the first-order condition in Eq. (S23). Note the perturbation $\hat{\rho}_1$ induces coherence between reflection-symmetric sites k and $L+1-k$, leading to the single-particle correlations (for $k \le L/2$)

$$\begin{aligned}
\langle \hat{b}^\dagger_k \hat{b}_{L+1-k}\rangle_\mathrm{w} &\approx \mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J}\,\mathrm{Tr}\big(\hat{b}^\dagger_k\hat{b}_{L+1-k}\,\hat{f}^\dagger_{L+1-k}\hat{f}_k\,\hat{\rho}_0\big)\\
&= \mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J}\,\mathrm{Tr}\Big[\hat{b}^\dagger_k\hat{b}_{L+1-k}\,\hat{b}^\dagger_{L+1-k}\hat{b}_k\prod_{i=k+1}^{L-k}(-1)^{\hat{n}_i}\hat{\rho}_0\Big] && \text{[using Eq. (S3)]}\\
&= \mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J}\,\mathrm{Tr}\Big[\hat{n}_k(1-\hat{n}_{L+1-k})\prod_{i=k+1}^{L-k}(-1)^{\hat{n}_i}\hat{\rho}_0\Big]\\
&= \mathrm{i}\,\frac{\gamma_+ + \gamma_-}{2J}\,\frac{\gamma_+}{\gamma_+ + \gamma_-}\,\frac{\gamma_-}{\gamma_+ + \gamma_-}\left(\frac{\gamma_- - \gamma_+}{\gamma_+ + \gamma_-}\right)^{L-2k} && \text{[using Eq. (S11)]}\\
&= \mathrm{i}\,\frac{\gamma_+\gamma_-}{2J(\gamma_+ + \gamma_-)}\left(\frac{\gamma_+ - \gamma_-}{\gamma_+ + \gamma_-}\right)^{L-2k} .
\end{aligned} \tag{S28}$$

Thus, the correlations fall off exponentially with distance due to the string, and vanish for $k < L/2$ if $\gamma_+ = \gamma_-$.

At strong dissipation ($\gamma_\pm \gg J$), the steady state approaches a step where all sites $i \le L/2$ are filled and all sites $i > L/2$ are empty (see Sec. V in the main text for a physical explanation). This pure state is expressed as

$$\hat{\rho}_\mathrm{step} = \hat{n}_1 \dots \hat{n}_{L/2}\,(1-\hat{n}_{L/2+1})\dots(1-\hat{n}_L)\ , \tag{S29}$$

and annihilated by the dissipators $\mathcal{D}[\hat{b}^\dagger_{L/2}]$ and $\mathcal{D}[\hat{b}_{L/2+1}]$ in Eq. (S22). To first order in $J/\gamma_\pm$, the steady state is of the form $\hat{\rho}_\mathrm{s} \approx \hat{\rho}_\mathrm{step} + \hat{\rho}_1$, such that

$$-(\mathrm{i}/\hbar)[\hat{H}, \hat{\rho}_\mathrm{step}] + \gamma_+\mathcal{D}[\hat{b}^\dagger_{L/2}]\hat{\rho}_1 + \gamma_-\mathcal{D}[\hat{b}_{L/2+1}]\hat{\rho}_1 = 0\ . \tag{S30}$$

We will show the perturbation $\hat{\rho}_1$ is again generated by the operator $\hat{Q}$ in Eq. (S25),

$$\hat{\rho}_1 = \mathrm{i}\,\frac{2J}{\gamma_+ + \gamma_-}\big(\hat{Q}\,\hat{\rho}_\mathrm{step} - \mathrm{H.c.}\big)\ . \tag{S31}$$

First, using the expression for $\hat{H}$ in Eq. (S4), one finds

$$\begin{aligned}
-(\mathrm{i}/\hbar)[\hat{H}, \hat{\rho}_\mathrm{step}] &= \mathrm{i}\,J\,\big[\hat{f}^\dagger_{L/2+1}\hat{f}_{L/2}, \hat{n}_1\dots\hat{n}_{L/2}\,(1-\hat{n}_{L/2+1})\dots(1-\hat{n}_L)\big] + \mathrm{H.c.}\\
&= \mathrm{i}\,J\hat{f}^\dagger_{L/2+1}\hat{f}_{L/2}\,\hat{\rho}_\mathrm{step} + \mathrm{H.c.}\ .
\end{aligned} \tag{S32}$$

Next, acting the dissipator $\mathcal{D}[\hat{b}^\dagger_{L/2}]$ on $\hat{\rho}_1$ and using $\hat{\rho}_\mathrm{step}\hat{b}_{L/2} = 0$ yields

$$\mathcal{D}[\hat{b}^\dagger_{L/2}]\hat{\rho}_1 = -\mathrm{i}\,\frac{J}{\gamma_+ + \gamma_-}\hat{f}_{L/2}\hat{f}^\dagger_{L/2}\hat{Q}\,\hat{\rho}_\mathrm{step} + \mathrm{H.c.} = -\mathrm{i}\,\frac{J}{\gamma_+ + \gamma_-}\hat{f}^\dagger_{L/2+1}\hat{f}_{L/2}\,\hat{\rho}_\mathrm{step} + \mathrm{H.c.}\ . \tag{S33}$$

The same result is found for $\mathcal{D}[\hat{b}_{L/2+1}]\hat{\rho}_1$. Thus,

$$\gamma_+\mathcal{D}[\hat{b}^\dagger_{L/2}]\hat{\rho}_1 + \gamma_-\mathcal{D}[\hat{b}_{L/2+1}]\hat{\rho}_1 = -\mathrm{i}\,J\hat{f}^\dagger_{L/2+1}\hat{f}_{L/2}\,\hat{\rho}_\mathrm{step} + \mathrm{H.c.}\ . \tag{S34}$$

This exactly cancels $-(\mathrm{i}/\hbar)[\hat{H}, \hat{\rho}_\mathrm{step}]$ in Eq. (S32), satisfying the first-order condition in Eq. (S30). This single-particle correlations can be calculated by a similar procedure as in Eq. (S28), yielding (for $k \le L/2$)

$$\begin{aligned}
\langle \hat{b}^\dagger_k\hat{b}_{L+1-k}\rangle_\mathrm{s} &\approx \mathrm{i}\,\frac{2J}{\gamma_+ + \gamma_-}\,\mathrm{Tr}\Big[\hat{n}_k(1-\hat{n}_{L+1-k})\prod_{i=k+1}^{L-k}(-1)^{\hat{n}_i}\hat{\rho}_\mathrm{step}\Big]\\
&= \mathrm{i}\,\frac{2J}{\gamma_+ + \gamma_-}(-1)^{L/2-k}\ .
\end{aligned} \tag{S35}$$

Now the string gives rise to constant-amplitude oscillations, stabilizing long-range coherence. Note the steady states in Eqs. (S24) and (S31) can be written as a compact matrix product operator, as in Refs. [S3, S6].

C. GEOMETRIC RESONANCE AND LONG-RANGE COHERENCE IN DIPOLE SETUPS

In the main text, we mentioned that the dissipation induced long-range coherence found above can be generalized to other "dipole" setups where the pump and loss act on neighboring sites ($q = p + 1$), provided p divides $L - p$, or vice versa. Here we present numerical examples. Figure S1 shows the single-particle density matrices in steady state for different dipole arrangements at strong dissipation. Here the pump and loss divide the system into weakly coupled segments, sites 1 through p and sites q through L, as explained in Sec. V of the main text. Long-range coherence is found when the two segments share a resonant mode, as sketched in the upper panels of Fig. S1. This is a higher-order analog of the geometric resonances discussed in the main text. In all of the examples, the coherences are limited to nearest neighbors at weak dissipation ($\gamma_\pm \ll J$). Hence, the resonant geometries constitute a family of setups where long-range coherence is stabilized by increasing the pump and loss rates.

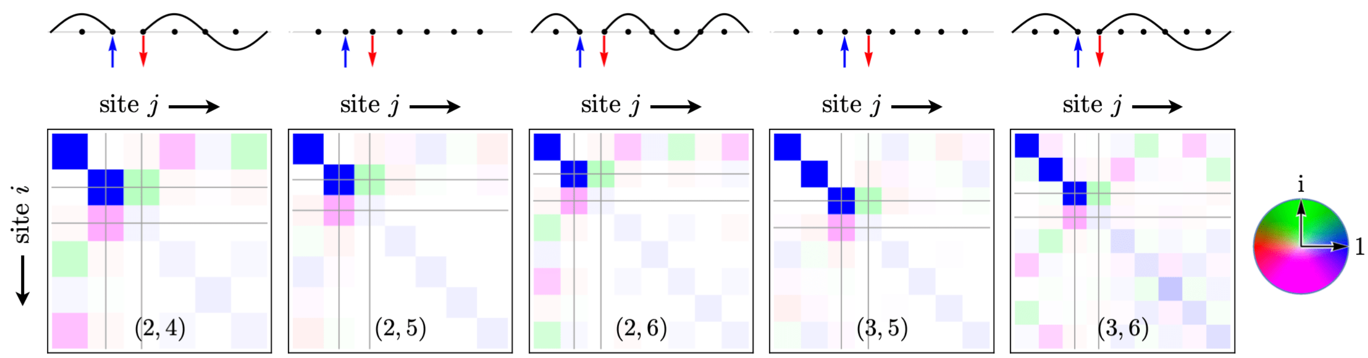

FIG. S1. Top panel shows schematic "dipole" setups where the pump and loss divide the system into two segments which may share a resonant mode, shown by black lines. Bottom panel shows steady-state correlations $\langle \hat{b}_i^\dagger \hat{b}_j \rangle$ for the corresponding setups at strong dissipation, $\gamma_+ = \gamma_- = 20J$. The numbers in parentheses give the size of the segments, p and $L - p$. The first four plots are obtained from exact diagonalization and the last plot is obtained by averaging over quantum trajectories [S7].

D. RATE EQUATIONS FOR WEAK DISSIPATION

In Sec. IV of the main text, we explained that when the dissipation is weak compared to the energy splitting of the single-particle modes, the modes become uncorrelated, leading to simplified rate equations for the mode occupations which characterize the steady state. Here we derive these approximate rate equations.

First, we consider the equation of motion for the expectation of a general observable $\hat{A}$. Substituting $\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\hat{\rho})$ into Eq. (S2) and using the cyclic property of trace, one finds

$$\frac{d\langle \hat{A} \rangle}{dt} = \mathrm{Tr}\left(\hat{A}\, \frac{d\hat{\rho}}{dt} \right) = \frac{\mathrm{i}}{\hbar} \langle [\hat{H}, \hat{A}] \rangle - \mathrm{Re} \sum\nolimits_{\alpha=\pm} \langle \hat{L}_\alpha^\dagger [\hat{L}_\alpha, \hat{A}] \rangle \, . \tag{S36}$$

The single-particle modes of the Hamiltonian in Eq. (S4) are of the form $\hat{F}_m = \sum_j c_{m,j} \hat{f}_j$, such that $\sum_m c_{m,i}^* c_{m,j} = \delta_{ij}$. One can invert this relation to find

$$\hat{f}_i = \sum\nolimits_{m=1}^{L} c_{m,i}^* \hat{F}_m \, . \tag{S37}$$

The number operator for a mode is given by $\hat{N}_m := \hat{F}_m^\dagger \hat{F}_m$ which, by definition, commutes with $\hat{H}$. The equation of motion for the mode occupation $N_m := \langle \hat{N}_m \rangle$ is simpler if one has pump and loss of free fermions, i.e., $\hat{L}_+ = \sqrt{\gamma_+}\, \hat{f}_p^\dagger$ and $\hat{L}_- = \sqrt{\gamma_-}\, \hat{f}_q$. Then Eq. (S36) yields

$$\dot{N}_m = -\gamma_+ \, \mathrm{Re}\, \langle \hat{f}_p [\hat{f}_p^\dagger, \hat{N}_m] \rangle - \gamma_- \, \mathrm{Re}\, \langle \hat{f}_q^\dagger [\hat{f}_q, \hat{N}_m] \rangle \, . \tag{S38}$$

The commutators can be found using Eq. (S37) and the relations $\{\hat{F}_m, \hat{F}_n\} = 0$ and $\{\hat{F}_m^\dagger, \hat{F}_n\} = \delta_{m,n}$, which gives

$$\langle \hat{f}_q^\dagger [\hat{f}_q, \hat{N}_m] \rangle = c_{m,q}^* \sum\nolimits_n c_{n,q} \langle \hat{F}_n^\dagger \hat{F}_m \rangle \, . \tag{S39}$$

Now we approximate the modes to be uncorrelated, i.e., $\langle \hat{F}_n^\dagger \hat{F}_m\rangle \approx \delta_{m,n} N_m$, obtaining $\langle \hat{f}_q^\dagger [\hat{f}_q, \hat{N}_m]\rangle \approx |c_{m,q}|^2 N_m$. The pump term in Eq. (S38) is found by exchanging $p \leftrightarrow q$ and particles with holes, yielding

$$\dot{N}_m \approx \gamma_+ |c_{m,p}|^2 \bar{N}_m - \gamma_- |c_{m,q}|^2 N_m \;, \tag{S40}$$

where $\bar{N}_m := 1 - N_m$ is the hole occupation.

For pump and loss of hard-core bosons, i.e., $\hat{L}_+ = \sqrt{\gamma_+}\,\hat{b}_p^\dagger$ and $\hat{L}_- = \sqrt{\gamma_-}\,\hat{b}_q$, Eq. (S38) is modified as

$$\dot{N}_m = -\gamma_+ \operatorname{Re}\langle \hat{b}_p [\hat{b}_p^\dagger, \hat{N}_m]\rangle - \gamma_- \operatorname{Re}\langle \hat{b}_q^\dagger [\hat{b}_q, \hat{N}_m]\rangle \;. \tag{S41}$$

Mapping the bosons onto fermions through Eq. (S3) and using the expansion in Eq. (S37), we find

$$\langle \hat{b}_q^\dagger [\hat{b}_q, \hat{N}_m]\rangle = c_{m,q}^* \sum\nolimits_n c_{n,q} \langle \hat{F}_n^\dagger \hat{F}_m\rangle + \sum\nolimits_{n,n'} c_{n,q} c_{n',q}^* \left\langle \hat{F}_n^\dagger (-1)^{\hat{\mathcal{N}}_q} \left[(-1)^{\hat{\mathcal{N}}_q}, \hat{N}_m\right] \hat{F}_{n'} \right\rangle \;, \tag{S42}$$

where $\hat{\mathcal{N}}_q := \sum_{i<q} \hat{n}_i$. The interaction can be simplified by writing $\hat{N}_m = \sum_{i,j} c_{m,i}^* c_{m,j} \hat{f}_i^\dagger \hat{f}_j$ and noting that $(-1)^{\hat{\mathcal{N}}_q}$ transforms $\hat{f}_i$ to $-\hat{f}_i$ only if $i < q$. Thus,

$$(-1)^{\hat{\mathcal{N}}_q} \hat{N}_m (-1)^{\hat{\mathcal{N}}_q} = \sum\nolimits_{i,j} \sigma_{i-q} \sigma_{j-q} c_{m,i}^* c_{m,j} \hat{f}_i^\dagger \hat{f}_j \;, \tag{S43}$$

where $\sigma_k = 1$ for $k \geq 0$ and -1 for $k < 0$. Rewriting the $\hat{f}_j$'s in terms of the modes in Eq. (S37), one finds

$$(-1)^{\hat{\mathcal{N}}_q} \hat{N}_m (-1)^{\hat{\mathcal{N}}_q} = \sum\nolimits_{r,s} \alpha_{m,r}^{(q)} \alpha_{s,m}^{(q)} \hat{F}_r^\dagger \hat{F}_s \;, \tag{S44}$$

$$\text{where} \quad \alpha_{m,n}^{(i)} := \sum\nolimits_j \sigma_{j-i} c_{m,j}^* c_{n,j} \;. \tag{S45}$$

Substituting Eq. (S44) into Eq. (S42) gives

$$\langle \hat{b}_q^\dagger [\hat{b}_q, \hat{N}_m]\rangle = c_{m,q}^* \sum\nolimits_n c_{n,q} \langle \hat{F}_n^\dagger \hat{F}_m\rangle + \sum\nolimits_{n,n'} c_{n,q} c_{n',q}^* \left[\langle \hat{F}_n^\dagger \hat{F}_m^\dagger \hat{F}_m \hat{F}_{n'}\rangle - \sum\nolimits_{r,s} \alpha_{m,r}^{(q)} \alpha_{s,m}^{(q)} \langle \hat{F}_n^\dagger \hat{F}_r^\dagger \hat{F}_s \hat{F}_{n'}\rangle\right] \;. \tag{S46}$$

Thus, the string gives rise to quartic coupling among the modes. Note the above result is exact for hard-core bosons. We approximate the quartic terms using the product-of-modes ansatz, $\hat{\rho} \approx N_m |1_m\rangle\langle 1_m| + \bar{N}_m\rangle |0_m\rangle\langle 0_m|$, presented in the main text (recall, $\bar{x} := 1 - x$), finding

$$\langle \hat{F}_m^\dagger \hat{F}_{m'}^\dagger \hat{F}_n \hat{F}_{n'}\rangle \approx N_m N_{m'} (\delta_{m,n'} \delta_{m',n} - \delta_{m,n} \delta_{m',n'}) \;, \tag{S47}$$

along with $\langle \hat{F}_n^\dagger \hat{F}_m\rangle \approx \delta_{m,n} N_m$. Using these expressions in Eq. (S46) yields

$$\langle \hat{b}_q^\dagger [\hat{b}_q, \hat{N}_m]\rangle \approx N_m \big(\bar{N}_m |c_{m,q}|^2 + n_q\big) - n_q \beta_m^{(q)} + |\kappa_m^{(q)}|^2 \;, \tag{S48}$$

where

$$\beta_m^{(i)} := \sum\nolimits_n N_n |\alpha_{m,n}^{(i)}|^2 \quad \text{and} \quad \kappa_m^{(i)} := \sum\nolimits_n N_n \alpha_{n,m}^{(i)} c_{n,i} \;. \tag{S49}$$

Note that n_q in Eq. (S48) is the occupation at the loss site, and not a mode index. It is related to the mode occupations as $n_q := \langle \hat{f}_q^\dagger \hat{f}_q\rangle \approx \sum_n N_n |c_{n,q}|^2$. The analog of Eq. (S48) for the pump term is again found by swapping the particle and hole occupations and exchanging $p \leftrightarrow q$, which gives

$$\langle \hat{b}_p [\hat{b}_p^\dagger, \hat{N}_m]\rangle \approx -\bar{N}_m \big(N_m |c_{m,p}|^2 + \bar{n}_p\big) + \bar{n}_p \bar{\beta}_m^{(p)} - |\kappa_m^{(p)} - c_{m,p}|^2 \;. \tag{S50}$$

For the last two terms, we have used $\sum_n |\alpha_{m,n}^{(i)}|^2 = 1$ and $\sum_n \alpha_{n,m}^{(i)} c_{n,i} = c_{m,i}$ in Eq. (S49). Substituting Eqs. (S48) and (S50) into Eq. (S41) yields the coupled nonlinear rate equations for hard-core bosons,

$$\dot{N}_m \approx \gamma_+ \big[\bar{N}_m \big(N_m |c_{m,p}|^2 + \bar{n}_p\big) - \bar{n}_p \bar{\beta}_m^{(p)} + |\kappa_m^{(p)} - c_{m,p}|^2\big] - \gamma_- \big[N_m \big(\bar{N}_m |c_{m,q}|^2 + n_q\big) - n_q \beta_m^{(q)} + |\kappa_m^{(q)}|^2\big] \;. \tag{S51}$$

Once the steady state is found by solving Eq. (S40) or (S51), the two-site correlations can be obtained by using the product-of-modes approximation [see Eq. (S47)],

$$\langle \hat{f}_i^\dagger \hat{f}_j\rangle = \sum\nolimits_{m,n} c_{m,i} c_{n,j}^* \langle \hat{F}_m^\dagger \hat{F}_n\rangle \approx \sum\nolimits_m N_m c_{m,i} c_{m,j}^* \;, \tag{S52a}$$

$$\text{and} \quad \langle \hat{f}_i^\dagger \hat{f}_j^\dagger \hat{f}_i \hat{f}_j\rangle = \sum\nolimits_{m,n,m',n'} c_{m,i} c_{n,j} c_{m',i}^* c_{n',j}^* \langle \hat{F}_m^\dagger \hat{F}_n^\dagger \hat{F}_{m'} \hat{F}_{n'}\rangle \approx |\langle \hat{f}_i^\dagger \hat{f}_j\rangle|^2 - n_i n_j \;. \tag{S52b}$$

The latter can be used to find the density-density correlations for both free fermions and hard-core bosons,

$$\langle \hat{n}_i \hat{n}_j\rangle - n_i n_j := \langle \hat{f}_i^\dagger \hat{f}_i \hat{f}_j^\dagger \hat{f}_j\rangle - n_i n_j \approx \delta_{ij} n_i - |\langle \hat{f}_i^\dagger \hat{f}_j\rangle|^2 \;. \tag{S53}$$

E. EFFECTIVE ZENO DYNAMICS AT STRONG DISSIPATION

In Sec. V of the main text, we modeled the dynamics at strong dissipation. Here, at long times, the pump and loss sites are pinned to occupation 1 and 0, respectively, dividing the system into weakly coupled segments. In particular, we discussed that the source (or sink) can be effectively replaced by a weak correlated pump (or loss) at its neighboring sites, which can lead to striking resonant features in steady state. Here we derive this effective Zeno dynamics using the formalism developed in Ref. [S8].

We first summarize the relevant findings in Ref. [S8]. Consider a system described by a Hamiltonian $\hat{H}$ and subject to strong dissipation characterized by a rate γ, that acts only on a subspace $\mathcal{H}_0$ of the full Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes \mathcal{H}_1$. The dissipator $\mathcal{L}_0$ targets a unique (mixed) state $\hat{\psi}_0$ in this subspace, i.e., $\mathcal{L}_0\hat{\psi}_0 = 0$. Then, at all times $t \gg 1/\gamma$, the density matrix $\hat{\rho}$ is well approximated as $\hat{\rho} \approx \hat{\psi}_0 \otimes \hat{\rho}_{\text{eff}}$, where $\hat{\rho}_{\text{eff}}$ encodes the state in $\mathcal{H}_1$. The time evolution of $\hat{\rho}_{\text{eff}}$ depends on the spectrum of $\mathcal{L}_0$, which is composed of eigenvalues ξ_ν, and left and right eigenvectors $\hat{\phi}_\nu$ and $\hat{\psi}_\nu$ such that $\text{Tr}_{\mathcal{H}_0}(\hat{\phi}_\mu\hat{\psi}_\nu) = \delta_{\mu,\nu}$ $(\mu,\nu \geq 0)$. In particular, one can show $d\hat{\rho}_{\text{eff}}/dt \approx -(\text{i}/\hbar)[\hat{H}_{\text{eff}}, \hat{\rho}_{\text{eff}}] + \mathcal{D}_{\text{eff}}\hat{\rho}_{\text{eff}}$ with

$$\hat{H}_{\text{eff}} = \hat{g}_0 + \sum\nolimits_{\mu,\nu>0} (\text{Im } Y_{\mu,\nu})\, \hat{g}_\mu^\dagger \hat{g}_\nu \;, \tag{S54}$$

$$\text{and} \quad \mathcal{D}_{\text{eff}}\, \hat{\rho}_{\text{eff}} = \sum\nolimits_{\mu,\nu>0} (\text{Re } Y_{\mu,\nu}) \left(2\hat{g}_\nu \hat{\rho}_{\text{eff}}\, \hat{g}_\mu^\dagger - \{\hat{g}_\mu^\dagger \hat{g}_\nu, \hat{\rho}_{\text{eff}}\}\right) \;, \tag{S55}$$

where the operators $\hat{g}_\nu$ and coefficients $Y_{\mu,\nu}$ are given by

$$\hat{g}_\nu := \text{Tr}_{\mathcal{H}_0}[(\hat{\psi}_\nu \otimes \hat{\mathbb{1}}_{\mathcal{H}_1})\hat{H}] \;, \tag{S56}$$

$$\text{and} \quad Y_{\mu,\nu} := -\text{Tr}_{\mathcal{H}_0}(\hat{\phi}_\mu^\dagger \hat{\phi}_\nu \hat{\psi}_0)/\xi_\mu^* \;. \tag{S57}$$

Note the eigenvalues ξ_ν scale as γ, so the effective dissipation $\mathcal{D}_{\text{eff}}$ falls off as $1/\gamma$. Also, $\hat{g}_0$ in Eq. (S54) is simply the Hamiltonian $\hat{H}$ projected onto the target state $\hat{\psi}_0$.

In our qubit array, $\mathcal{L}_0 = \gamma_+ \mathcal{D}[\hat{b}_p^\dagger] + \gamma_- \mathcal{D}[\hat{b}_q]$, where $\mathcal{D}[\hat{b}_p^\dagger]$ and $\mathcal{D}[\hat{b}_q]$ are the pump and loss dissipators, respectively. (Recall, $\mathcal{D}[\hat{x}]\hat{\rho} := \hat{x}\hat{\rho}\hat{x}^\dagger - \{\hat{x}^\dagger\hat{x}, \hat{\rho}\}/2$.) As these two act on disjoint subspaces $(q > p)$, $\mathcal{L}_0$ can be diagonalized in terms of their individual eigenvalues and eigenvectors listed in Table I. The eigenvalues of $\mathcal{L}_0$ are given by $\xi_{\nu,\nu'} = \gamma_+\xi_\nu^p + \gamma_-\xi_{\nu'}^q$, and the corresponding eigenvectors are $\hat{\psi}_{\nu,\nu'} = \hat{\psi}_\nu^p \otimes \hat{\psi}_{\nu'}^q$ and $\hat{\phi}_{\nu,\nu'} = \hat{\phi}_\nu^p \otimes \hat{\phi}_{\nu'}^q$. One can readily verify the eigenvectors are normalized such that $\text{Tr}_{\mathcal{H}_p}(\hat{\phi}_\mu^p \hat{\psi}_\nu^p) = \delta_{\mu,\nu}$ and $\text{Tr}_{\mathcal{H}_q}(\hat{\phi}_\mu^q \hat{\psi}_\nu^q) = \delta_{\mu,\nu}$. As expected, the target state $\hat{\psi}_{0,0}$ describes a filled pump site and an empty loss site in the subspace $\mathcal{H}_0 = \mathcal{H}_p \otimes \mathcal{H}_q$. Using Eq. (S56), we find

$$\hat{g}_{0,0} = -J\sum\nolimits_{i=1}^{L-1} \text{Tr}_{\mathcal{H}_0}\left[\hat{b}_p^\dagger \hat{b}_p \hat{b}_q \hat{b}_q^\dagger \big(\hat{b}_{i+1}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+1}\big)\right] = -J\Big(\sum\nolimits_{i=1}^{p-2} + \sum\nolimits_{i=p+1}^{q-2} + \sum\nolimits_{i=q+1}^{L-1}\Big)\big(\hat{b}_{i+1}^\dagger \hat{b}_i + \text{H.c.}\big) \;. \tag{S58}$$

Thus, the Hamiltonian projected onto the Zeno subspace simply describes hopping in three uncoupled segments. The only nonzero coefficients $Y_{(\mu,\mu'),(\nu,\nu')}$ in Eq. (S57), with $\xi_{\mu,\mu'}, \xi_{\nu,\nu'} \neq 0$, are

$$Y_{(1,0),(1,0)} = 2/\gamma_+ \;, \quad Y_{(0,1),(0,1)} = 2/\gamma_- \;, \quad \text{and} \quad Y_{(1,1),(1,1)} = 2/(\gamma_+ + \gamma_-) \;. \tag{S59}$$

The corresponding dissipators $\hat{g}_{\nu,\nu'}$ in Eq. (S56) are obtained as (for $q > p$)

$$\hat{g}_{1,0} = -J\sum\nolimits_{i=1}^{L-1} \text{Tr}_{\mathcal{H}_0}\left[\hat{b}_p^\dagger \hat{b}_q \hat{b}_q^\dagger \big(\hat{b}_{i+1}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+1}\big)\right] = -J\left[(1-\delta_{p,1})\,\hat{b}_{p-1}^\dagger + (1-\delta_{q,p+1})\,\hat{b}_{p+1}^\dagger\right] \;, \tag{S60a}$$

$$\hat{g}_{0,1} = -J\sum\nolimits_{i=1}^{L-1} \text{Tr}_{\mathcal{H}_0}\left[\hat{b}_p^\dagger \hat{b}_p \hat{b}_q \big(\hat{b}_{i+1}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+1}\big)\right] = -J\left[(1-\delta_{q,p+1})\,\hat{b}_{q-1} + (1-\delta_{q,L})\,\hat{b}_{q+1}\right] \;, \tag{S60b}$$

$$\hat{g}_{1,1} = -J\sum\nolimits_{i=1}^{L-1} \text{Tr}_{\mathcal{H}_0}\left[\hat{b}_p^\dagger \hat{b}_q \big(\hat{b}_{i+1}^\dagger \hat{b}_i + \hat{b}_i^\dagger \hat{b}_{i+1}\big)\right] = -J\delta_{q,p+1}\hat{\mathbb{1}}_{\mathcal{H}_1} \;. \tag{S60c}$$

TABLE I. Eigenvalues $\xi_\nu^{p,q}$, right eigenvectors $\hat{\psi}_\nu^{p,q}$, and left eigenvectors $\hat{\phi}_\nu^{p,q}$ of the pump and loss dissipators $\mathcal{D}[\hat{b}_p^\dagger]$ and $\mathcal{D}[\hat{b}_q]$.

	ξ_ν^p	$\hat{\psi}_\nu^p$	$\hat{\phi}_\nu^p$	ξ_ν^q	$\hat{\psi}_\nu^q$	$\hat{\phi}_\nu^q$
$\nu = 0$	0	$\hat{n}_p$	$\hat{\mathbb{1}}_p$	0	$\hat{\mathbb{1}}_q - \hat{n}_q$	$\hat{\mathbb{1}}_q$
$\nu = 1$	$-1/2$	$\hat{b}_p^\dagger$	$\hat{b}_p$	$-1/2$	$\hat{b}_q$	$\hat{b}_q^\dagger$
$\nu = 2$	$-1/2$	$\hat{b}_p$	$\hat{b}_p^\dagger$	$-1/2$	$\hat{b}_q^\dagger$	$\hat{b}_q$
$\nu = 3$	-1	$\hat{\mathbb{1}}_p - 2\hat{n}_p$	$\hat{\mathbb{1}}_p - \hat{n}_p$	-1	$2\hat{n}_q - \hat{\mathbb{1}}_q$	$\hat{n}_q$

Substituting these results into Eqs. (S54) and (S55), one finds $\hat{H}_{\rm eff} = \hat{g}_{0,0}$ and $\mathcal{D}_{\rm eff} = \mathcal{D}[\hat{L}_+^{\rm eff}] + \mathcal{D}[\hat{L}_-^{\rm eff}]$, where

$$\hat{L}_+^{\rm eff} = \sqrt{\Gamma_+}\left[(1-\delta_{p,1})\,\hat{b}_{p-1}^\dagger + (1-\delta_{q,p+1})\,\hat{b}_{p+1}^\dagger\right] , \tag{S61a}$$

$$\hat{L}_-^{\rm eff} = \sqrt{\Gamma_-}\left[(1-\delta_{q,p+1})\,\hat{b}_{q-1} + (1-\delta_{q,L})\,\hat{b}_{q+1}\right] , \tag{S61b}$$

with rates $\Gamma_\pm := 4J^2/\gamma_\pm$. Therefore, the source and sink generates correlation between its neighboring sites through a second-order process, dissipatively coupling the segments in Eq. (S58). For pump and loss of free fermions instead of hard-core bosons, one finds the same expressions with $\hat{b}$'s replaced by $\hat{f}$'s in Eqs. (S61).

F. RATE EQUATIONS FOR STRONG DISSIPATION

We showed above that the qubit array reduces to weakly coupled segments at strong dissipation. Since this coupling is small compared to tunneling, any off-resonant modes become uncorrelated at long times, to a good approximation, as discussed in Sec. V of the main article. This is similar to what happens at weak dissipation (see Sec. D). However, the modes that are resonant in neighboring segments can remain coherent, producing surprising steady states. Here we derive the approximate rate equations that explain these features.

The pump and loss divide the system into three segments as in Eq. (S58). The single-particle modes in segment ν, with L_ν sites, have the form $\hat{F}^{(\nu)}_{m_\nu} = \sum_{j_\nu=1}^{L_\nu} c^{(\nu)}_{m_\nu,j_\nu} \hat{f}^{(\nu)}_{j_\nu}$, $m_\nu = 1,\ldots,L_\nu$. Here

$$\hat{f}^{(1)}_{j_1} := \hat{f}_{j_1} , \quad \hat{f}^{(2)}_{j_2} := \hat{f}_{p+j_2} , \quad \text{and} \quad \hat{f}^{(3)}_{j_3} := \hat{f}_{q+j_3} . \tag{S62}$$

Note, in this section we use ν to label the segments, and not eigenvalues as in the last section! The modes in neighboring segments are coupled through the dissipators in Eqs. (S61). For simplicity, we assume all three segments are present, i.e., $1<p<q-1<L-1$, which gives

$$\hat{L}_+^{\rm eff} = \sqrt{\Gamma_+}(\hat{b}_{p-1}^\dagger + \hat{b}_{p+1}^\dagger) \quad \text{and} \quad \hat{L}_-^{\rm eff} = \sqrt{\Gamma_-}(\hat{b}_{q-1} + \hat{b}_{q+1}) . \tag{S63}$$

To find the equations of motion, it is useful to write the dissipators in terms of the modes. To this end, we first invert the unitary coefficients $c^{(\nu)}_{m_\nu,j_\nu}$ to find $\hat{f}^{(\nu)}_{j_\nu}$, then attach appropriate string operators according to Eq. (S3), yielding

$$\hat{b}_{p-1} = (-1)^{\hat{\mathcal{N}}_1} \sum\nolimits_{m_1} v^{(1)*}_{m_1} \hat{F}^{(1)}_{m_1} , \qquad \hat{b}_{q-1} = -(-1)^{\hat{\mathcal{N}}_{1,2}} \sum\nolimits_{m_2} v^{(2)*}_{m_2} \hat{F}^{(2)}_{m_2} , \tag{S64a}$$

$$\hat{b}_{p+1} = -(-1)^{\hat{\mathcal{N}}_1} \sum\nolimits_{m_2} u^{(2)*}_{m_2} \hat{F}^{(2)}_{m_2} , \qquad \hat{b}_{q+1} = -(-1)^{\hat{\mathcal{N}}_{1,2}} \sum\nolimits_{m_3} u^{(3)*}_{m_3} \hat{F}^{(3)}_{m_3} , \tag{S64b}$$

where $u^{(\nu)}_{m_\nu} := c^{(\nu)}_{m_\nu,1}$, $v^{(\nu)}_{m_\nu} := c^{(\nu)}_{m_\nu,L_\nu}$, $\mathcal{N}_1$ is the total occupation in the first segment, and $\mathcal{N}_{1,2}$ is the total occupation in the first two segments. The extra minus signs arise from the filled pump site and do not affect the physics. Note the convention for $\mathcal{N}_1$ is different from that used in Sec. D. As explained in the main text, the steady state is set by the occupations $N^{(\nu)}_{m_\nu} := \langle \hat{N}^{(\nu)}_{m_\nu}\rangle = \langle \hat{F}^{(\nu)\dagger}_{m_\nu} \hat{F}^{(\nu)}_{m_\nu}\rangle$ and the correlations $\langle \hat{F}^{(\nu)\dagger}_{m_\nu} \hat{F}^{(\nu+1)}_{m_{\nu+1}}\rangle$. We find the rate equations for $N^{(3)}_{m_3}$ and $\langle \hat{F}^{(2)\dagger}_{m_2} \hat{F}^{(3)}_{m_3}\rangle$ from first principles, and later work out those for the other segments by symmetry.

The equation of motion for the occupations can be found from Eq. (S36),

$$\dot{N}^{(3)}_{m_3} = -\mathrm{Re}\,\langle \hat{L}_+^{\rm eff\dagger}[\hat{L}_+^{\rm eff}, \hat{N}^{(3)}_{m_3}]\rangle - \mathrm{Re}\,\langle \hat{L}_-^{\rm eff\dagger}[\hat{L}_-^{\rm eff}, \hat{N}^{(3)}_{m_3}]\rangle . \tag{S65}$$

Using Eqs. (S64), one finds $[\hat{L}_+^{\rm eff}, \hat{N}^{(3)}_{m_3}] = [\hat{b}_{q-1}, \hat{N}^{(3)}_{m_3}] = 0$ and $[\hat{b}_{q+1}, \hat{N}^{(3)}_{m_3}] = -(-1)^{\hat{\mathcal{N}}_{1,2}} u^{(3)*}_{m_3} \hat{F}^{(3)}_{m_3}$, which give

$$\dot{N}^{(3)}_{m_3} = -\Gamma_-\,\mathrm{Re}\,u^{(3)*}_{m_3}\left[\sum\nolimits_{m_2} v^{(2)}_{m_2}\langle \hat{F}^{(2)\dagger}_{m_2}\hat{F}^{(3)}_{m_3}\rangle + \sum\nolimits_{m_3'} u^{(3)}_{m_3'}\langle \hat{F}^{(3)\dagger}_{m_3'}\hat{F}^{(3)}_{m_3}\rangle\right] . \tag{S66}$$

Next, we make the approximation that all off-resonant modes are uncorrelated and that the spectrum is nondegenerate in a given segment, i.e., $\langle \hat{F}^{(\nu)\dagger}_{m_\nu}\hat{F}^{(\nu)}_{m_\nu'}\rangle \approx \delta_{m_\nu,m_\nu'} N^{(\nu)}_{m_\nu}$ as in Sec. D, and $\langle \hat{F}^{(\nu)\dagger}_{m_\nu}\hat{F}^{(\nu+1)}_{m_{\nu+1}}\rangle \approx \Lambda^{(\nu,\nu+1)}_{m_\nu,m_{\nu+1}} T^{(\nu,\nu+1)}_{m_\nu,m_{\nu+1}}$, where $\Lambda^{(\nu,\nu')}_{m_\nu,m_{\nu'}}$ is 1 for resonant modes and 0 otherwise. Substituting these into Eq. (S66) yields

$$\dot{N}^{(3)}_{m_3} \approx -\Gamma_-\left[|u^{(3)}_{m_3}|^2 N^{(3)}_{m_3} + \mathrm{Re}\sum\nolimits_{m_2}\Lambda^{(2,3)}_{m_2,m_3} v^{(2)}_{m_2} u^{(3)*}_{m_3} T^{(2,3)}_{m_2,m_3}\right] . \tag{S67}$$

The same rate equation is obtained if the hard-core bosons were replaced by free fermions in Eq. (S63).

To find the rate equation for the correlations, we first generalize Eq. (S36) for a non-Hermitian operator $\hat{X}$. Using $\langle \hat{X}\rangle = \mathrm{Tr}(\hat{X}\hat{\rho})$ in Eq. (S2), along with the effective Hamiltonians and dissipators, one finds

$$\frac{d\langle \hat{X}\rangle}{dt} = \frac{\mathrm{i}}{\hbar}\langle[\hat{H}_{\rm eff}, \hat{X}]\rangle + \frac{1}{2}\sum\nolimits_{\alpha=\pm}\langle[\hat{L}_\alpha^{\rm eff\dagger}, \hat{X}]\hat{L}_\alpha^{\rm eff}\rangle - \langle \hat{L}_\alpha^{\rm eff\dagger}[\hat{L}_\alpha^{\rm eff}, \hat{X}]\rangle . \tag{S68}$$

We consider the correlation between two resonant modes m_2 and m_3, i.e., $\hat{X} = \hat{F}^{(2)\dagger}_{m_2}\hat{F}^{(3)}_{m_3}$ with $\Lambda^{(2,3)}_{m_2,m_3} = 1$. Since the two modes have equal energies, $[\hat{H}_{\rm eff}, \hat{X}] = 0$. Further, using Eqs. (S63) and (S64) gives $[\hat{L}^{\rm eff}_+, \hat{X}] = [\hat{b}_{p-1}, \hat{X}] = 0$ and $[\hat{b}_{p+1}, \hat{X}] = -(-1)^{\hat{\mathcal{N}}_1} u^{(2)*}_{m_2}\hat{F}^{(3)}_{m_3}$. Noting that $\{(-1)^{\hat{\mathcal{N}}_1}, \hat{F}^{(1)}_{m_1}\} = 0$, and $[(-1)^{\hat{\mathcal{N}}_1}, \hat{F}^{(\nu)}_{m_\nu}] = 0$ for $\nu = 2, 3$, we obtain

$$\begin{aligned}\langle[\hat{L}^{\rm eff\dagger}_+, \hat{X}]\hat{L}^{\rm eff}_+\rangle - \langle\hat{L}^{\rm eff\dagger}_+[\hat{L}^{\rm eff}_+, \hat{X}]\rangle &= -\Gamma_+ u^{(2)*}_{m_2}\Big[\sum\nolimits_{m_1} v^{(1)}_{m_1}\langle\hat{F}^{(1)\dagger}_{m_1}\hat{F}^{(3)}_{m_3}\rangle + \sum\nolimits_{m_2'} u^{(2)}_{m_2'}\langle\hat{F}^{(2)\dagger}_{m_2'}\hat{F}^{(3)}_{m_3}\rangle\Big] \\ &\approx -\Gamma_+|u^{(2)}_{m_2}|^2\, T^{(2,3)}_{m_2,m_3}\ . \end{aligned} \tag{S69}$$

Similarly, for the loss dissipator, we use $[(-1)^{\hat{\mathcal{N}}_{1,2}}, \hat{F}^{(3)}_{m_3}] = 0$, and $\{(-1)^{\hat{\mathcal{N}}_{1,2}}, \hat{F}^{(\nu)}_{m_\nu}\} = 0$ for $\nu = 1, 2$, which gives

$$[\hat{L}^{\rm eff}_-, \hat{X}] = -\sqrt{\Gamma_-}(-1)^{\hat{\mathcal{N}}_{1,2}}\big[v^{(2)*}_{m_2}\hat{F}^{(3)}_{m_3} + 2\hat{X}\big(\hat{f}_{q-1} + \hat{f}_{q+1}\big)\big]\ , \tag{S70a}$$

$$[\hat{L}^{\rm eff\dagger}_-, \hat{X}] = \sqrt{\Gamma_-}\big[u^{(3)}_{m_3}\hat{F}^{(2)\dagger}_{m_2} + 2\big(\hat{f}^\dagger_{q-1} + \hat{f}^\dagger_{q+1}\big)\hat{X}\big](-1)^{\hat{\mathcal{N}}_{1,2}}\ . \tag{S70b}$$

Combining these results with Eqs. (S64) yields

$$\langle[\hat{L}^{\rm eff\dagger}_-, \hat{X}]\hat{L}^{\rm eff}_-\rangle - \langle\hat{L}^{\rm eff\dagger}_-[\hat{L}^{\rm eff}_-, \hat{X}]\rangle \approx -\Gamma_-\big[\big(|v^{(2)}_{m_2}|^2 + |u^{(3)}_{m_3}|^2\big)T^{(2,3)}_{m_2,m_3} + v^{(2)*}_{m_2}u^{(3)}_{m_3}\big(N^{(2)}_{m_2} + N^{(3)}_{m_3}\big)\big] + 4\langle\hat{L}^{\rm eff\dagger}_-\hat{X}\hat{L}^{\rm eff}_-\rangle\ . \tag{S71}$$

Substituting Eqs. (S69) and (S71) into Eq. (S68), we find the rate equation

$$\dot{T}^{(2,3)}_{m_2,m_3} \approx -\Gamma_f\, T^{(2,3)}_{m_2,m_3} + 2\,\zeta_b\,\langle\hat{L}^{\rm eff\dagger}_-\hat{F}^{(2)\dagger}_{m_2}\hat{F}^{(3)}_{m_3}\hat{L}^{\rm eff}_-\rangle - \Gamma_- v^{(2)*}_{m_2}u^{(3)}_{m_3}\big[N^{(2)}_{m_2} + N^{(3)}_{m_3}\big]/2\ , \tag{S72}$$

where $\Gamma_f := \frac{\Gamma_+}{2}|u^{(2)}_{m_2}|^2 + \frac{\Gamma_-}{2}\big[|v^{(2)}_{m_2}|^2 + |u^{(3)}_{m_3}|^2\big]$, and $\zeta_b = 1$ for the hard-core bosons we have been considering. For free fermion dissipators, one obtains the same equation without the quartic term, i.e., $\zeta_b = 0$.

The quartic term can be approximated by pairwise contractions similar to Eq. (S47), yielding

$$\langle\hat{L}^{\rm eff\dagger}_-\hat{F}^{(2)\dagger}_{m_2}\hat{F}^{(3)}_{m_3}\hat{L}^{\rm eff}_-\rangle \approx -\langle\hat{L}^{\rm eff\dagger}_-\hat{L}^{\rm eff}_-\rangle\, T^{(2,3)}_{m_2,m_3} + \Gamma_-\big\langle\hat{F}^{(2)\dagger}_{m_2}\big(\hat{f}_{q-1} + \hat{f}_{q+1}\big)\big\rangle\big\langle\big(\hat{f}^\dagger_{q-1} + \hat{f}^\dagger_{q+1}\big)\hat{F}^{(3)}_{m_3}\big\rangle\ . \tag{S73}$$

Using this result in Eq. (S72), we find the decay rate of correlations is effectively enhanced for the hard-core bosons,

$$\Gamma_b \approx \Gamma_f + 2\langle\hat{L}^{\rm eff\dagger}_-\hat{L}^{\rm eff}_-\rangle\ , \tag{S74}$$

which dampens the resonant features in steady state. When the correlations are small compared to 1, the dominant correction in Eq. (S73) comes from this decay rate, $\langle\hat{L}^{\rm eff\dagger}_-\hat{L}^{\rm eff}_-\rangle \approx \Gamma_- n_{q-1} \sim O(\Gamma_-)$. On the other hand, $\Gamma_f \sim O(\Gamma_\pm/l)$, where $l = \min(L_2, L_3)$. Thus, for large systems, $\Gamma_b/\Gamma_f \sim O(l)$.

The rate equations for the other segments can be found by symmetry from Eqs. (S67) and (S72). In particular, we exchange segments $1 \leftrightarrow 3$, rates $\Gamma_+ \leftrightarrow \Gamma_-$, amplitudes $u \leftrightarrow v^*$, and particles with holes ($\hat{F} \leftrightarrow \hat{F}^\dagger$) to obtain

$$\dot{N}^{(1)}_{m_1} \approx \Gamma_+\Big[|v^{(1)}_{m_1}|^2\bar{N}^{(1)}_{m_1} - {\rm Re}\sum\nolimits_{m_2}\Lambda^{(1,2)}_{m_1,m_2}v^{(1)}_{m_1}u^{(2)*}_{m_2}T^{(1,2)}_{m_1,m_2}\Big]\ , \tag{S75}$$

$$\text{and}\quad \dot{T}^{(1,2)}_{m_1,m_2} \approx -\Gamma_f'\, T^{(1,2)}_{m_1,m_2} + 2\,\zeta_b\,\langle\hat{L}^{\rm eff\dagger}_+\hat{F}^{(1)\dagger}_{m_1}\hat{F}^{(2)}_{m_2}\hat{L}^{\rm eff}_+\rangle + \Gamma_+ v^{(1)*}_{m_1}u^{(2)}_{m_2}\big[\bar{N}^{(1)}_{m_1} + \bar{N}^{(2)}_{m_2}\big]/2\ , \tag{S76}$$

where $\Gamma_f' := \frac{\Gamma_+}{2}\big[|v^{(1)}_{m_1}|^2 + |u^{(2)}_{m_2}|^2\big] + \frac{\Gamma_-}{2}|v^{(2)}_{m_2}|^2$, and recall that $\bar{x} := 1 - x$. The quartic term can again be approximated by pairwise contractions, yielding an increased decay rate for the bosons, $\Gamma_b' \approx \Gamma_f' + 2\langle\hat{L}^{\rm eff\dagger}_+\hat{L}^{\rm eff}_+\rangle \approx \Gamma_+\bar{n}_{p+1}$. The rate equation for $\hat{N}^{(2)}_{m_2}$ is found by combining those for $\hat{N}^{(1)}_{m_1}$ and $\hat{N}^{(3)}_{m_3}$, and making suitable exchanges, which give

$$\begin{aligned}\dot{N}^{(2)}_{m_2} \approx\ & \Gamma_+\Big[|u^{(2)}_{m_2}|^2\bar{N}^{(2)}_{m_2} - {\rm Re}\sum\nolimits_{m_1}\Lambda^{(1,2)}_{m_1,m_2}v^{(1)}_{m_1}u^{(2)*}_{m_2}T^{(1,2)}_{m_1,m_2}\Big] \\ & - \Gamma_-\Big[|v^{(2)}_{m_2}|^2 N^{(2)}_{m_2} + {\rm Re}\sum\nolimits_{m_3}\Lambda^{(2,3)}_{m_2,m_3}v^{(2)}_{m_2}u^{(3)*}_{m_3}T^{(2,3)}_{m_2,m_3}\Big]\ .\end{aligned} \tag{S77}$$

Equations (S67), (S75), and (S77) show that, without any resonance ($\Lambda = 0$), $N^{(1)}_{m_1} \to 1$ and $N^{(3)}_{m_3} \to 0$ in steady state, while $\hat{N}^{(2)}_{m_2}$ approaches a fraction. When resonances are present, these can be dramatically altered by the coupling to the coherences in Eqs. (S72) and (S76), producing surprising density-wave order as shown in the main text.

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